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Weighted approximation of functions on the real line by Bernstein polynomials

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Abstract

The authors give error estimates, a Voronovskaya-type relation, strong converse results and saturation for the weighted approximation of functions on the real line with Freud weights by Bernstein-type operators.

Keywords: Freud weights; Weighted modulus of smoothness and K-functional; Bernstein operator

1. Introduction

In Chlodovsky [1] introduced some Bernstein-type polynomial operators to approximate unbounded functions on the real line, but he gave only pointwise convergence results without error estimates. For practical purposes discrete linear operators are more useful than continuous operators (like convolution integrals, etc.). However, the divergence behaviour of weighted Lagrange interpolation with

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Freud weights for continuous functions on the real line is well-known (see [8]). Also Hermite–Fejér interpolation does not solve the weighted approximation problem on the real line, because the weight in the corresponding error estimate is different from the weight in the definition of the function class (see [8,9]).

Recently the authors in [2] introduced some Bernstein-type operators for the weighted approximation of functions on $[-1,1]$ with endpoint or inner singularities and they showed direct and converse results. In this respect, we mention the recent paper [5] where weighted approximation by Bernstein polynomials on $[-1,1]$ is considered. However, in [5] only continuous functions are examined (without singularities), and thus the use of weights is not properly justified.

In this paper, we construct a Bernstein-type operator for the weighted approximation of functions on $(-\infty,+\infty)$ with respect to Freud weights and give error estimates, a Voronovskaya-type relation, strong converse results and solve the saturation problem (see Theorems 1–5).

2. Main results

In the following c denotes a positive constant which may assume different values in different formulas. Moreover, let $v\sim\mu$, for v and μ two quantities depending on some parameters, if $|v/\mu|^{\pm 1}\leqslant c$, with c independent of the parameters.

Let

$$w(x)=e^{-Q(x)},\quad x\in(-\infty,+\infty)$$

be a Freud weight, with $Q(x)$ satisfying the following conditions:

(a) $Q\in C^2(0,+\infty)$ is a positive even function,
(b) $\lim_{x\to\infty} x\frac{Q''(x)}{Q'(x)}=\gamma>0$,
(c) if $\gamma=1$ or 3, then Q'' is nondecreasing

(see [4, Definition 11.3.1, p. 184]). Evidently, condition (b) implies that for sufficiently large x both $Q'(x)$ and $Q''(x)$ are positive, and $Q'(x)$ tends to infinity as x does.

Now consider the class of functions

$$C_w=\{f\in C(\mathbf{R}):\lim_{|x|\to\infty}(wf)(x)=0\}$$

equipped with the norm $\|wf\|_{C_w}:=\|wf\|=\sup_x|(wf(x)|$. We also put $\|wf\|_{[c,d]}=\max_{c\leqslant x\leqslant d}|(wf)(x)|$. For $f\in C_w$ the weighted modulus of smoothness is

$$\omega_2(f,t)_w=\sup_{0<h\leqslant t}\|w\Delta_h^2 f\|_{[-h^*,h^*]}+\inf_{\ell\in\mathscr{P}_1}\|w(f-\ell)\|_{[t^*,\infty)}+\inf_{\ell\in\mathscr{P}_1}\|w(f-\ell)\|_{(-\infty,-t^*]},\tag{1}$$

where t^* is defined by $tQ'(t^*)=1$ (see [4, Definition 11.2.2, p. 182]) and $\mathscr{P}_n$, $n\in\mathbb{N}$, is the set of algebraic polynomials of degree at most n.

Next, we define a sequence of positive real numbers $\{\lambda_n\}$ by

$$\lambda_n Q'(\lambda_n) = \sqrt{n}, \quad n > n_0 \tag{2}$$

with n_0 sufficiently large. For $n \leqslant n_0$, let $\lambda_n = \lambda_{n_0}$; with this extension of the definition, the entire sequence $\{\lambda_n\}_1^\infty$ will be monotone increasing.

Note that these numbers are in close relation with the so-called Mhaskar–Rahmanov–Saff numbers a_n with respect to the weight w which play a crucial role in the infinite–finite range inequalities for polynomials. In fact, it is easy to see that

$$\lambda_n \sim a_{\sqrt{n}}$$

(cf. e.g. [6, (5.3) and (5.5)]).

For every $f \in C_w$ let

$$B_n(f, x) = \sum_{k=0}^{n} p_{n,k}(x) f(x_k) \tag{3}$$

with

$$p_{n,k}(x) = \frac{1}{2^n}\binom{n}{k}\left(1 + \frac{x}{2\lambda_n}\right)^k \left(1 - \frac{x}{2\lambda_n}\right)^{n-k}, \quad x_k = x_{k,n} = 2\lambda_n \frac{2k - n}{n} \tag{4}$$

and finally our Bernstein-type operator is

$$B_n^*(f, x) = \begin{cases} B_n(f, x) & \text{if } |x| \leqslant \lambda_n, \\ B_n(f, \lambda_n) + B_n'(f, \lambda_n)(x - \lambda_n) & \text{if } x \geqslant \lambda_n, \\ B_n(f, -\lambda_n) + B_n'(f, -\lambda_n)(x + \lambda_n) & \text{if } x \leqslant -\lambda_n. \end{cases} \tag{5}$$

Remark. Note that $B_n^{*'} \in \mathrm{AC}_{\mathrm{loc}}$ and B_n^* is a linear operator, which reproduces linear functions ℓ, i.e., $B_n^*(\ell, x) \equiv \ell(x)$. We could not consider only B_n because its weighted norm is not bounded.

We have the following error estimate.

Theorem 1. *If $f \in C_w$, then*

$$\|w(f - B_n^*(f))\| \leqslant c\omega_2\left(f, \frac{\lambda_n}{\sqrt{n}}\right)_w. \tag{6}$$

We remark that (2) implies $\lambda_n = o(\sqrt{n})$, i.e. estimate (6) yields convergence.

Next, we state an asymptotic relation of Voronovskaya-type for the operator B_n^*.

Theorem 2. *Assume $f \in C_w$ such that $f''(x)$ exists at a fixed point x. Then*

$$\lim_{n \to \infty} \frac{n}{\lambda_n^2}[f(x) - B_n^*(f, x)] = -2f''(x). \tag{7}$$

Remark. Comparing this with the Voronovskaya relation for the classical Bernstein operator on $[0,1]$, we see that the factor $1-x^2$ on the right-hand side in (7) is missing here, as expected.

We also state a strong converse result.

Theorem 3. *We have*

$$\|w[f-B_n^*(f)]\| = O\left(\frac{\lambda_n}{\sqrt{n}}\right)^{\alpha} \quad \Leftrightarrow \quad \omega_2(f,t)_w \leqslant ct^{\alpha}, \quad 0<\alpha<2.$$

The following result yields the trivial class of saturation.

Theorem 4. *We have*

$$\|w[f-B_n^*(f)]\| = o\left(\frac{\lambda_n^2}{n}\right) \ \Leftrightarrow \ f \ \textit{is linear}.$$

Finally we settle the problem of saturation class.

Theorem 5. *We have*

$$\|w[f-B_n^*(f)]\| = O\left(\frac{\lambda_n^2}{n}\right) \ \Leftrightarrow \ \omega_2(f,t)_w \leqslant ct^2.$$

3. Proof of Theorem 1

The proof of Theorem 1 is based on several lemmas. First we state a property of Freud weights which is folklore, but for completeness we include a proof.

Lemma 1. *For Freud weights $w(x) = e^{-Q(x)}$ we have*

$$Q'(ax) \leqslant AQ'(x), \quad a \geqslant 1, \ x \geqslant x_0, \tag{8}$$

where the constant $A>1$ depends only on a.

Proof. Let first $a = a_0 = e^{\frac{1}{2(\gamma+1)}}$. Then by $Q''(x)>0$ for $x \geqslant x_0$ and property (b) of Freud weights we get

$$\begin{aligned} Q'(a_0x) - Q'(x) &= \int_x^{a_0x} Q''(t)\,dt \leqslant (\gamma+1)\int_x^{a_0x} \frac{Q'(t)}{t}\,dt \\ &\leqslant (\gamma+1)Q'(a_0x)\log a_0 = \frac{1}{2}Q'(a_0x), \quad x \geqslant x_0, \end{aligned}$$

whence

$$Q'(a_0x) \leqslant 2Q'(x), \quad x \geqslant x_0 \tag{9}$$

which proves the lemma for $a = a_0$.

Now let $a \geqslant 1$ be arbitrary, $a_0^{k-1} \leqslant a < a_0^k$, say. Then using (9) repeatedly we obtain

$$\begin{aligned} Q'(ax) &\leqslant Q'(a_0^k x) \leqslant 2Q'(a_0^{k-1}x) \leqslant 2^2 Q'(a_0^{k-2}x) \\ &\leqslant \cdots \leqslant 2^k Q'(x) \leqslant 2^{2(\gamma+1)\log a+1} Q'(x), \quad x \geqslant x_0. \qquad \square \end{aligned}$$

The boundedness of the weighted norm of our operator will be proved separately for the interval

$$I_n := [-\lambda_n, \lambda_n]$$

and for its complement. In fact, the weighted boundedness of B_n will be proved for the larger interval

$$J_n := \left[-\frac{3}{2}\lambda_n, \frac{3}{2}\lambda_n\right]$$

for later purposes.

Lemma 2. *Let $f \in C_w$. Then*

$$\|wB_n(f)\|_{J_n} \leqslant c\|wf\|. \tag{10}$$

Proof. Let $x \in J_n$ and assume n even (in the case n odd we need only a technical modification). Then

$$w(x)B_n(f, x) = \sum_{j=-n/2}^{n/2} p_{n,n/2+j}(x) f(y_j) w(y_j) e^{Q(y_j)-Q(x)}$$

with

$$y_j = 4j\lambda_n/n, \quad j = 0, \pm 1, \ldots, \pm n/2.$$

Let $0 \leqslant x \leqslant \frac{3}{2}\lambda_n$. (The case x negative is similar.) Then

$$w(x)|B_n(f, x)| \leqslant \|wf\| \sum_{j=-n/2}^{n/2} p_{n,n/2+j}(x) e^{Q(y_j)-Q(x)} := A\|wf\|.$$

We prove that A is bounded. Let us consider the partition

$$A = \left\{ \sum_{|y_j| \leqslant x} + \sum_{|y_j| > x} \right\} p_{n,n/2+j}(x) e^{Q(y_j)-Q(x)} := \Sigma_1 + \Sigma_2. \tag{11}$$

Since $\Sigma_1 \leqslant 1$, it is sufficient to estimate Σ_2. Here we may assume that $0 \leqslant j \leqslant n/2$, since $p_{n/2-j}(x) \leqslant p_{n/2+j}(x)$ for these values of j (because of the assumption $0 \leqslant x \leqslant 3\lambda_n/2$ and the evenness of Q). When $j = n/2$, by $Q'(2\lambda_n) \leqslant cQ'(\lambda_n) = c\sqrt{n}/\lambda_n$ (this follows from Lemma 1), it is easy to see that

$$\left(\frac{2\lambda_n + x}{4\lambda_n}\right)^n e^{Q(2\lambda_n)-Q(x)} \leqslant \left(\frac{2\lambda_n + x}{4\lambda_n}\right)^n e^{Q'(2\lambda_n)(2\lambda_n - x)}$$
$$\leqslant \left(1 - \frac{2\lambda_n - x}{4\lambda_n}\right)^n e^{c\sqrt{n}(2\lambda_n - x)/\lambda_n}, \quad 0 \leqslant x \leqslant 3\lambda_n/2.$$

In the given interval considered the latter function is monotone increasing and thus attains its maximum $(\frac{7}{8})^n e^{c\sqrt{n}/2} = o(1)$ at $x = \frac{3}{2}\lambda_n$.

Now let $0 \leqslant j < n/2$. By Stirling formula we deduce

$$p_{n,n/2+j}(x) \leqslant \frac{c}{2^n} \frac{(n/e)^n \sqrt{n}(1 + \frac{x}{2\lambda_n})^{n/2+j}(1 - \frac{x}{2\lambda_n})^{n/2-j}}{(\frac{n/2+j}{e})^{n/2+j}\sqrt{\frac{n^2}{4} - j^2}(\frac{n/2-j}{e})^{n/2-j}}$$
$$= \frac{2c\lambda_n}{\sqrt{n(4\lambda_n^2 - y_j^2)}}\left(\frac{2\lambda_n + x}{2\lambda_n + y_j}\right)^{\frac{n}{4\lambda_n}(2\lambda_n + y_j)}\left(\frac{2\lambda_n - x}{2\lambda_n - y_j}\right)^{\frac{n}{4\lambda_n}(2\lambda_n - y_j)}$$
$$= \frac{2c\lambda_n}{\sqrt{n(4\lambda_n^2 - y_j^2)}} \exp\left\{\frac{n}{4\lambda_n}\left[(2\lambda_n + y_j)\log\left(1 - \frac{y_j - x}{2\lambda_n + y_j}\right)\right.\right.$$
$$\left.\left. + (2\lambda_n - y_j)\log\left(1 + \frac{y_j - x}{2\lambda_n - y_j}\right)\right]\right\}, \quad 0 \leqslant x \leqslant 3\lambda_n/2. \tag{12}$$

Here we distinguish two cases.

Case 1: $x \leqslant y_j \leqslant \lambda_n + x/2 (\leqslant \frac{7}{4}\lambda_n)$. Then using the inequality

$$\log(1 + u) \leqslant u - \frac{u^2}{2} + \frac{u^3}{3}, \quad u > -1 \tag{13}$$

to estimate the log terms in (12) we obtain

$$p_{n,n/2+j}(x) \leqslant \frac{c}{\sqrt{n}} \exp\left\{\frac{n}{2}\left[-\frac{(y_j - x)^2}{4\lambda_n^2 - y_j^2} + \frac{8y_j(y_j - x)^3}{3(4\lambda_n^2 - y_j^2)^2}\right]\right\}$$
$$\leqslant \frac{c}{\sqrt{n}} \exp\left\{\frac{n}{2}\left[-\frac{(y_j - x)^2}{4\lambda_n^2 - y_j^2} + \frac{8\,\frac{2\lambda_n + x}{2}\,\frac{2\lambda_n - x}{2}\,(y_j - x)^2}{3[4\lambda_n^2 - (\frac{2\lambda_n + x}{2})^2](4\lambda_n^2 - y_j^2)}\right]\right\}$$
$$\leqslant \frac{c}{\sqrt{n}} \exp\left\{-\frac{cn(y_j - x)^2}{\lambda_n^2}\left[1 - \frac{8(2\lambda_n + x)}{3(6\lambda_n + x)}\right]\right\}$$
$$\leqslant \frac{c}{\sqrt{n}} \exp\left\{-\frac{cn(y_j - x)^2(2\lambda_n - x)}{\lambda_n^2(6\lambda_n + x)}\right\}$$
$$\leqslant \frac{c}{\sqrt{n}} \exp\left\{-\frac{cn(y_j - x)^2}{\lambda_n^2}\right\}, \quad 0 \leqslant x \leqslant 3\lambda_n/2.$$

On the other hand

$$Q(y_j) - Q(x) \leqslant (y_j - x)Q'(y_j) \leqslant (y_j - x)Q'(2\lambda_n)$$
$$\leqslant c\,\frac{\sqrt{n}(y_j - x)}{\lambda_n}, \quad 0 \leqslant x \leqslant 3\lambda_n/2 \tag{14}$$

and thus

$$\sum_{x \leqslant y_j \leqslant \lambda_n + x/2} p_{n,n/2+j}(x) e^{Q(y_j)-Q(x)}$$
$$\leqslant \frac{c}{\sqrt{n}} \sum_{j=0}^{n/2} \exp\left\{ -\frac{c_1 n (y_j - x)^2}{\lambda_n^2} + \frac{c_2 \sqrt{n}(y_j - x)}{\lambda_n} \right\}$$
$$\leqslant \frac{c}{\sqrt{n}} \sum_{j=0}^{\infty} e^{-c_1 \frac{j^2}{n} + c_2 \frac{j}{\sqrt{n}}} \leqslant \frac{c}{\sqrt{n}} \left(\sum_{j \leqslant \frac{2c_2}{c_1}\sqrt{n}} e^{2c_2^2/c_1} + \sum_{j \geqslant \frac{2c_2}{c_1}\sqrt{n}} e^{-\frac{c_2 j}{\sqrt{n}}} \right) \leqslant c.$$

Case 2: $\lambda_n + x/2 < y_j \leqslant (1 - \frac{2}{n})2\lambda_n$. Then estimating only the first logarithm in (12) by $\log(1+u) < u$, we get

$$p_{n,n/2+j}(x) \leqslant c \exp\left\{ \frac{n}{4\lambda_n} \left[-(y_j - x) + (2\lambda_n - y_j) \log \frac{2\lambda_n - x}{2\lambda_n - y_j} \right] \right\},$$
$$0 \leqslant x \leqslant 3\lambda_n/2.$$

Here the function $\varphi(u) := -u + (2\lambda_n - u)\log\frac{2\lambda_n - x}{2\lambda_n - u}$ is monotone decreasing in the interval $[x, 2\lambda_n]$, whence putting $y_j = \lambda_n + x/2$ we get

$$p_{n,n/2+j}(x) \leqslant c \exp\left\{ \frac{n}{4\lambda_n} [-(\lambda_n - x/2) + (\lambda_n - x/2)\log 2] \right\}$$
$$\leqslant c \exp\left[-\frac{cn(2\lambda_n - x)}{\lambda_n} \right] \leqslant e^{-cn}, \quad 0 \leqslant x \leqslant 3\lambda_n/2.$$

On the other hand, (14) yields $Q(y_j) - Q(x) \leqslant c\sqrt{n}$, and thus

$$\sum_{\lambda_n + x/2 < y_j \leqslant (1-\frac{2}{n})2\lambda_n} p_{n,n/2+j}(x) e^{Q(y_j)-Q(x)} \leqslant e^{c\sqrt{n}} \sum_{j=1}^{n} e^{-cn} = o(1). \qquad \square$$

Lemma 3. *We have*

$$\|wB_n''(f)\|_{J_n} \leqslant c\,\frac{n}{\lambda_n^2}\|wf\| \quad \textit{if} \quad f \in C_w \tag{15}$$

and

$$\|wB_n''(f)\|_{J_n} \leqslant c\|wf''\| \quad \textit{if} \quad f'' \in C_w. \tag{16}$$

Proof. By linear transformations (see [3]) we obtain

$$B_n''(f,x) = \frac{n^2}{(4\lambda_n^2 - x^2)^2} \sum_{k=0}^{n} p_{n,k}(x)(x_k - x)^2 f(x_k) + \frac{2nx}{(4\lambda_n^2 - x^2)^2} \sum_{k=0}^{n} p_{n,k}(x)(x_k - x) f(x_k) - \frac{n}{(4\lambda_n^2 - x^2)} B_n(f,x).$$

Therefore by Lemma 2 (used for the last term)

$$w(x)|B_n''(f,x)| \leqslant c\|wf\| \left\{ \frac{n^2}{\lambda_n^4} \sum_{k=0}^{n} p_{n,k}(x)(x_k - x)^2 e^{Q(x_k)-Q(x)} + \frac{n}{\lambda_n^3} \sum_{k=0}^{n} p_{n,k}(x)|x_k - x| e^{Q(x_k)-Q(x)} + \frac{n}{\lambda_n^2} \right\}, \quad x \in J_n.$$

Then by Cauchy–Schwarz inequality it follows that

$$w(x)|B_n''(f,x)| \leqslant c\|wf\| \left\{ \frac{n^2}{\lambda_n^4} \left[\sum_{k=0}^{n} p_{n,k}(x)(x_k - x)^4 \sum_{k=0}^{n} p_{n,k}(x) e^{2Q(x_{k,n})-2Q(x)} \right]^{1/2} + \frac{n}{\lambda_n^3} \left[\sum_{k=0}^{n} p_{n,k}(x)(x_k - x)^2 \sum_{k=0}^{n} p_{n,k}(x) e^{2Q(x_k)-2Q(x)} \right]^{1/2} + \frac{n}{\lambda_n^2} \right\}, \quad x \in J_n.$$

By a linear transformation, it follows from well-known inequalities that

$$\sum_{k=0}^{n} p_{n,k}(x)(x_k - x)^{2i} \leqslant c \left(\frac{\lambda_n^2}{n} \right)^i, \quad x \in J_n, \quad i = 0, 1, 2, \ldots \tag{17}$$

(cf. [7, inequality (6) on p. 15]). Hence Lemma 2 used with w^2 in place of w,

$$\|w(x)B_n''(f,x)\|_{J_n} \leqslant c\|wf\| \left(\frac{n^2}{\lambda_n^4} \frac{\lambda_n^2}{n} + \frac{n}{\lambda_n^3} \frac{\lambda_n}{\sqrt{n}} + \frac{n}{\lambda_n^2} \right) \leqslant c \frac{n}{\lambda_n^2} \|wf\|.$$

Now we prove (16). (In this part of the proof we have to use the more precise notation $x_{k,n}$ instead of x_k; see (4).) By [7, 1.4 (2)], we get for $f'' \in C_w$

$$w(x)|B_n''(f,x)| \leqslant c \frac{n^2}{\lambda_n^2} \sum_{k=0}^{n-2} |\Delta_{4\lambda_n/n}^2 f(x_{k+1,n})| p_{n-2,k}(x) w(x) \leqslant c \sum_{k=0}^{n-2} |f''(\xi_k)| w(\xi_k) p_{n-2,k}(x) e^{Q(\xi_k)-Q(x)},$$

where $\xi_k \in (x_{k,n}, x_{k+2,n})$. Here, if $|\xi_k| \leqslant |x_{k,n-2}|$ then $Q(\xi_k) \leqslant Q(x_{k,n-2})$. Otherwise, using that λ_n/n is a monotone decreasing sequence (see (2)) we get

$$|x_{k,n-2}| < |\xi_k| \leqslant \max|x_{k+1\pm1,n}| = \frac{2\lambda_n}{n}\max|2(k\pm1)-(n-2)|$$
$$\leqslant \frac{2\lambda_{n-2}}{n-2}\max|2(k\pm1)-(n-2)| = \max|x_{k\pm1,n-2}| \leqslant |x_{k,n-2}| + \frac{2\lambda_{n-2}}{n-2},$$

whence

$$Q(\xi_k) \leqslant Q(x_{k,n-2}) + Q'(3\lambda_{n-2})\frac{2\lambda_{n-2}}{n-2} \leqslant Q(x_{k,n-2}) + \frac{c}{\sqrt{n}}.$$

Thus using the boundedness of (11) with $n-2$ instead of n

$$w(x)|B_n''(f,x)| \leqslant c||wf''||\sum_{k=0}^{n-2} p_{n-2,k}(x)e^{Q(x_{k,n-2})-Q(x)} \leqslant c||wf''||$$

for $x \in J_{n-2}$. Now this is easily extended to $x \in J_n$, since the $\frac{3}{2}$ in the definition of J_n can be replaced by any $1 < \mu < 2$. □

We now complete the proof of boundedness of the operator on the whole real line.

Lemma 4. *We have*

$$||wB_n^*(f)|| \leqslant c||wf||$$

for all $f \in C_w$.

Proof. Because of Lemma 2 we may assume e.g. $x \geqslant \lambda_n$. We get by (5)

$$\begin{aligned} w(x)|B_n^*(f,x)| &= w(x)|B_n(f,\lambda_n) + B_n'(f,\lambda_n)(x-\lambda_n)| \\ &\leqslant c||wf|| + (x-\lambda_n)e^{Q(\lambda_n)-Q(x)}w(\lambda_n)|B_n'(f,\lambda_n)| \\ &\leqslant c||wf|| + (x-\lambda_n)e^{(\lambda_n-x)Q'(\lambda_n)}w(\lambda_n)|B_n'(f,\lambda_n)| \\ &\leqslant c||wf|| + (x-\lambda_n)e^{(\lambda_n-x)\sqrt{n}/\lambda_n}w(\lambda_n)|B_n'(f,\lambda_n)| \\ &\leqslant c||wf|| + \frac{\lambda_n}{\sqrt{n}}w(\lambda_n)|B_n'(f,\lambda_n)|. \end{aligned} \tag{18}$$

Here we use the inequality (2.2.14) from [4], as well as (10) and (15):

$$\begin{aligned} w(\lambda_n)|B_n'(f,\lambda_n)| &\leqslant c\left[\frac{\sqrt{n}}{\lambda_n}w(\xi_n)|B_n(f,\xi_n)| + \frac{\lambda_n}{\sqrt{n}}w(\eta_n)|B_n''(f,\eta_n)|\right] \\ &\leqslant c\frac{\sqrt{n}}{\lambda_n}||wf|| \end{aligned} \tag{19}$$

with suitable $\xi_n, \eta_n \in [\lambda_n - \frac{\lambda_n}{\sqrt{n}}, \lambda_n]$ (namely, on this interval evidently $w(\xi_n) \sim w(\eta_n) \sim w(\lambda_n)$). Substituting these estimates into (18) we get

$$w(x)|B_n^*(f,x)| \leqslant c||wf||\left(1 + \frac{\lambda_n}{\sqrt{n}}\frac{\sqrt{n}}{\lambda_n}\right) \leqslant c||wf||. \qquad \square$$

Proof of Theorem 1. By Lemma 4

$$||w(f - B_n^*(f)) \leqslant ||w(f-g)|| + ||w(g - B_n^*(g)|| \\ + ||wB_n^*(f-g)|| \leqslant c||w(f-g)|| + ||w(g - B_n^*(g))||, \tag{20}$$

where $g'' \in C_w$ will be chosen later.

Let first $x \in J_n$. Since

$$g(x) - g(x_k) = -g'(x)(x - x_k) - \frac{g''(\xi_k)}{2}(x - x_k)^2, \quad \xi_k \in (x, x_k),$$

and B_n reproduces linear functions, we can write

$$g(x) - B_n(g,x) = -\sum_{k=0}^{n} p_{n,k}(x) g''(\xi_k)(x - x_k)^2.$$

Therefore

$$w(x)|g(x) - B_n(g,x)| \leqslant ||wg''|| \sum_{k=0}^{n} p_{n,k}(x)(x - x_k)^2 e^{Q(\xi_k) - Q(x)} \\ \leqslant c\frac{\lambda_n^2}{n}||wg''||, \quad x \in J_n. \tag{21}$$

The last inequality can be seen similarly as in the proof of Lemma 3.

Let now, e.g. $x \geqslant \lambda_n$. Let $\ell(x)$ be that linear function which realizes the first infimum in (1) with respect to g and for $t = \lambda_n/\sqrt{n}$ (which implies $t^* = \lambda_n$ by (2)). Then by (5) and(21) applied to $g - \ell$ in place of g for $x = \lambda_n \in J_n$ we get

$$\begin{aligned} & w(x)|g(x) - B_n^*(g,x)| \\ &= w(x)\left|g(x) - \sum_{i=0}^{1} \frac{B_n^{(i)}(g, \lambda_n)}{i!}(x - \lambda_n)^i\right| \\ &\leqslant w(x)|g(x) - \ell(x)| + w(x)\sum_{i=0}^{1} \frac{|B_n^{(i)}(g - \ell, \lambda_n)|}{i!}(x - \lambda_n)^i \\ &\leqslant c\left[\omega_2\left(g, \frac{\lambda_n}{\sqrt{n}}\right)_w + c\frac{\lambda_n^2}{n}||wg''|| + \frac{\lambda_n}{\sqrt{n}} w(\lambda_n)|B_n'(g - \ell, \lambda_n)|\right], \quad x \geqslant \lambda_n. \end{aligned} \tag{22}$$

For the K-functional

$$K_2(f, t^2)_w := \inf_{g' \in \mathrm{AC_{loc}}} [||w[f - g]|| + t^2||wg''||] \tag{23}$$

we have the following equivalence relation:

$$K_2(f,t^2)_w \sim \omega_2(f,t)_w \tag{24}$$

(cf. Theorem 11.2.3 in [4, p. 182]). Hence if $f' \in \mathrm{AC}_{\mathrm{loc}}$, then $\omega_2(f,t)_w \leqslant ct^2\|wf''\|$. Using this with g instead of f we get from (22)

$$w(x)|g(x) - B_n^*(g,x)| \leqslant \frac{\lambda_n}{\sqrt{n}} w(\lambda_n)|B_n'(g-\ell,\lambda_n)| + c\frac{\lambda_n^2}{n}\|wg''\|, \quad x \geqslant \lambda_n. \tag{25}$$

Using the analogue of (19) with $g-\ell$ in place of f we get

$$w(\lambda_n)|B_n'(g-\ell,\lambda_n)| \leqslant c\left[\frac{\sqrt{n}}{\lambda_n} w(\xi_n)|B_n(g-\ell,\xi_n)| + \frac{\lambda_n}{\sqrt{n}} w(\eta_n)|B_n''(g-\ell,\eta_n)|\right] \tag{26}$$

with suitable $\xi_n, \eta_n \in [\lambda_n, \lambda_n + \frac{\lambda_n}{\sqrt{n}}] \subset J_n$. Here, by (21)

$$\begin{aligned} w(\xi_n)|B_n(g-\ell,\xi_n)| &\leqslant c\frac{\lambda_n^2}{n}\|wg''\| + w(\xi_n)|g(\xi_n) - \ell(\xi_n)| \\ &\leqslant c\frac{\lambda_n^2}{n}\|wg''\| + \omega_2\left(g, \frac{\lambda_n}{\sqrt{n}}\right)_w \leqslant c\frac{\lambda_n^2}{n}\|wg''\| \end{aligned}$$

and by (16)

$$w(\eta_n)|B_n''(g-\ell,\eta_n)| \leqslant c\|wg''\|.$$

Substituting these estimates into (26),

$$w(\lambda_n)|B_n'(g-\ell,\lambda_n)| \leqslant c\frac{\lambda_n}{\sqrt{n}}\|wg''\|.$$

This together with (21) yields from (25)

$$\|w(g - B_n^*(g)\| \leqslant c\frac{\lambda_n^2}{n}\|wg''\|.$$

Thus (20) gives

$$\|w(f - B_n^*(f))\| \leqslant c\left[\|w(f-g)\| + \frac{\lambda_n^2}{n}\|wg''\|\right].$$

One can see from the proof of the inequality $K_2(f,t^2) \leqslant c\omega_2(f,t)_w$ on pp. 191–192 of [4] (which is part of the equivalence relation (24)) that the function g in (23) can be chosen such that $g'' \in C_w$ (and not just $g' \in \mathrm{AC}_{\mathrm{loc}}$). With this choice we obtain

$$\|w(f - B_n^{\star}(f))\| \leqslant cK_2\left(f, \frac{\lambda_n^2}{n}\right)_w \leqslant c\omega_2\left(f, \frac{\lambda_n}{\sqrt{n}}\right)_w. \qquad \square \tag{27}$$

4. Proof of Theorem 2

Let $x \geqslant 0$ be fixed and define

$$\eta_x(\xi) := 2\frac{f(\xi) - f(x) + f'(x)(x - \xi)}{(x - \xi)^2} - f''(x). \tag{28}$$

Using this function η_x, for sufficiently large n we can write

$$\begin{aligned} &f(x) - B_n^*(f, x) \\ &\quad = f(x) - B_n(f, x) = \sum_{k=0}^{n} p_{n,k}(x)[f(x) - f(x_k)] \\ &\quad = -\frac{f''(x)}{2}\sum_{k=0}^{n} p_{n,k}(x)(x - x_k)^2 - \frac{1}{2}\sum_{k=0}^{n} p_{n,k}(x)(x - x_k)^2\eta_x(x_k), \end{aligned} \tag{29}$$

since B_n reproduces linear functions. Here

$$\begin{aligned} \sum_{k=0}^{n} p_{n,k}(x)(x - x_k)^2|\eta_x(x_k)| &\leqslant \left\{\sum_{|x_k - x| \leqslant \delta} + \sum_{|x_k - x| > \delta}\right\} p_{n,k}(x)(x - x_k)^2|\eta_x(x_k)| \\ &:= A_1 + A_2. \end{aligned} \tag{30}$$

Evidently $\lim_{\xi \to x} \eta_x(\xi) = 0$, i.e. for any $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|\eta_x(\xi)| < \varepsilon \quad \text{if} \quad |x - \xi| < \delta. \tag{31}$$

Next,

$$\sum_{k=0}^{n} p_{n,k}(x)(x - x_k)^2 := \frac{4\lambda_n^2 - x^2}{n} \tag{32}$$

(this comes from a linear transformation of a well-known formula, cf. [7, p. 14]). From (31) and (32) we deduce

$$A_1 \leqslant \varepsilon \sum_k p_{n,k}(x)(x - x_k)^2 \leqslant \frac{4\lambda_n^2}{n}\varepsilon. \tag{33}$$

On the other hand

$$\begin{aligned} |\eta_x(x_k)| &\leqslant \frac{c + |f(x_k)| + c|x_k|}{\delta^2} + c \leqslant \frac{c + \|wf\| \exp Q(x_k) + c\lambda_n}{\delta^2} \leqslant \frac{c\sqrt{n}}{\delta^2} e^{Q(x_k)} \\ &\leqslant \frac{c(x)\sqrt{n}}{\delta^2} e^{|x - x_k| Q'(2\lambda_n)} \leqslant \frac{c(x)\sqrt{n}}{\delta^2} e^{|x - x_k|\sqrt{n}/\lambda_n} \end{aligned}$$

for $|x - x_k| > \delta$ and for sufficiently large n's, where $c(x)$ depends only on x. Now it follows from the proof of Lemma 2 that

$$\frac{n}{\lambda_n^2} A_2 \leqslant \frac{c(x)n}{\lambda_n^2 \delta^2} \sum_{|x-x_k|>\delta} \exp\left\{ -\frac{c_1 n}{\lambda_n^2}(x-x_k)^2 + \frac{c_2\sqrt{n}}{\lambda_n}|x-x_k| \right\}$$
$$\leqslant \frac{c(x)n^2}{\delta^2} \exp\left\{ -\frac{c_1 n}{\lambda_n^2}\delta^2 + \frac{c_2\sqrt{n}}{\lambda_n}\delta \right\} \leqslant \frac{c(x)n^2}{\delta^2} \exp\left\{ -\frac{c_3 n}{\lambda_n^2}\delta^2 \right\}$$

for sufficiently large ns. Condition (b) in the definition of the Freud weights implies that $Q'(x) \geqslant cx^{\gamma-\varepsilon}$ for large x and for any $\varepsilon > 0$, whence by (2) $\lambda_n \leqslant cn^{\frac{1}{2(1+\gamma-\varepsilon)}}$. Thus we obtain

$$\frac{n}{\lambda_n^2} A_2 \leqslant \frac{c(x)n^2}{\delta^2} \exp\left\{ -\frac{c_1 n^{\frac{\gamma-\varepsilon}{1+\gamma-\varepsilon}}}{\delta^2} \right\} \to 0 \quad \text{as} \quad n \to \infty. \tag{34}$$

Thus by (30)–(34)

$$\left| \frac{n}{\lambda_n^2}[f(x) - B_n^*(f,x)] + 2f''(x) \right| \leqslant c\varepsilon + \frac{n}{\lambda_n^2} A_2 + \frac{x^2|f''(x)|}{2\lambda_n^2}.$$

Hence the theorem follows. $\square$

5. Proof of Theorem 3

If $\omega_2(f,t)_w \leqslant ct^\alpha$, $0 < \alpha < 2$, then by Theorem 1 we have

$$||w(f - B_n^*(f))|| \leqslant C\left(\frac{\lambda_n}{\sqrt{n}}\right)^\alpha. \tag{35}$$

Now we prove the converse implication. Assuming (35), by Lemma 3 we obtain

$$\Psi(n) := K_2(f, n^{-2})_w \leqslant c\{||w[f - B_n^*(f)]|| + n^{-2}||wB_n^{*''}(f)||\}$$
$$\leqslant c\left\{ \left(\frac{\lambda_n}{\sqrt{n}}\right)^\alpha + \frac{1}{n^2}[||wB_n^{*''}(f-g)|| + ||wB_n^{*''}(g)||] \right\}$$
$$\leqslant c\left\{ \left(\frac{\lambda_n}{\sqrt{n}}\right)^\alpha + \frac{1}{n\lambda_n^2}\left[||w(f-g)|| + \frac{\lambda_n^2}{n}||wg''|| \right] \right\}.$$

Thus with a proper choice of $g' \in \mathrm{AC}_{\mathrm{loc}}$,

$$\begin{aligned}\Psi(n) &\leqslant c\left\{\left(\frac{\lambda_n}{\sqrt{n}}\right)^{\alpha}+\frac{1}{n\lambda_n^2}K_2\left(f,\frac{\lambda_n^2}{n}\right)_w\right\}\\ &\leqslant c\left\{\left(\frac{\lambda_n}{\sqrt{n}}\right)^{\alpha}+\frac{1}{n\lambda_n^2}K_2\left(f,\frac{1}{[\sqrt{n}/\lambda_n]^2}\right)_w\right\}\\ &\leqslant c\left\{[\sqrt{n}/\lambda_n]^{-\alpha}+\left(\frac{[\sqrt{n}/\lambda_n]}{n}\right)^2\Psi([\sqrt{n}/\lambda_n])\right\}.\end{aligned}$$

Hence by the Berens–Lorentz lemma [4, Lemma 9.3.4, p. 122][2] we get

$$\omega_2(f,n^{-1})_w \leqslant cK_2(f,n^{-2})_w = \Psi(n) \leqslant cn^{-\alpha}.$$

This is equivalent to $\omega_2(f,t)_w \leqslant ct^{\alpha}$. □

6. Proof of Theorem 4

If f is a linear function, by Theorem 1 we get $B_n^*(f) \equiv f$. Now we prove the converse implication.

Let $[a,b]$ be an arbitrary but fixed interval (since n is large enough, we can always assume $[a,b] \subseteq [-\lambda_n,\lambda_n]$). Then following [3, 5.3, p. 124] we introduce the function

$$F(x) = f(x) - \frac{f(b)-f(a)}{b-a}(x-a) - f(a).$$

Note that $F(a) = F(b) = 0$.

We want to prove that $F \equiv 0$ on $[a,b]$, i.e., f is a linear function. To this aim suppose that there exists a $\xi \in (a,b)$, such that $F(\xi) > 0$. We will show that this leads to a contradiction, that is F must be identically 0.

We apply a slight modification of the parabola technique. By [3, Lemma 5.1, p. 124] there exists an $\eta \in (a,b)$ and

$$P(x) = \alpha(x-\eta)^2 + \beta(x-\eta) + F(\eta),$$

such that $\alpha < 0$ and $P(x) \geqslant F(x),\ x \in [a,b]$. Note that $P(\eta) = F(\eta)$. Let $\delta = \min(\eta - a, b-\eta) > 0$. Then

$$\begin{aligned}B_n(F,\eta) - B_n(P,\eta) = B_n(F-P,\eta) &= \sum_{k=0}^{n}[F(x_k)-P(x_k)]p_{n,k}(\eta)\\ &= \left\{\sum_{|x_k-\eta|\leqslant\delta} + \sum_{|x_k-\eta|>\delta}\right\}[F(x_k)-P(x_k)]p_{n,k}(\eta)\\ &:= S_1 + S_2. \qquad (36)\end{aligned}$$

[2] Note that at the quoted place this lemma is erroneously stated: $An^{-\alpha}$ in the condition should be $Ak^{-\alpha}$.

Evidently $S_1 \leqslant 0$, and thus

$$B_n(F,\eta) - B_n(P,\eta) \leqslant \sum_{|x_k-\eta|>\delta} [|F(x_k)| + |P(x_k)|] p_{n,k}(\eta). \tag{37}$$

From the definition of F, for $|\eta - x_k| > \delta$ we get

$$|F(x_k)| \leqslant |f(x_k)| + c|x_k| + d \leqslant ||wf|| e^{Q(x_k)} + c\lambda_n \leqslant e^{c|\eta - x_k|\sqrt{n}/\lambda_n}. \tag{38}$$

Moreover from the definition of P

$$|P(x_k)| \leqslant c x_k^2 < c\lambda_n^2 < e^{c|\eta - x_k|\sqrt{n}/\lambda_n}. \tag{39}$$

Hence and by (37)–(39)

$$B_n(F,\eta) - B_n(P,\eta) \leqslant \sum_{|x_k-\eta|>\delta} p_{n,k}(\eta) e^{c|\eta - x_k|\sqrt{n}/\lambda_n}$$

and following the proof of Lemma 2 we get

$$B_n(F,\eta) - B_n(P,\eta) \leqslant n \exp\left(c_1 \delta \frac{\sqrt{n}}{\lambda_n} - c_2 \frac{n}{\lambda_n^2} \delta^2 \right) \leqslant \exp\left(-c_3 \delta^2 \frac{n}{\lambda_n^2} \right).$$

Therefore

$$\begin{aligned} B_n(F,\eta) &\leqslant B_n(P,\eta) + \exp\left(-c_3 \delta^2 \frac{n}{\lambda_n^2} \right) \\ &= \alpha B_n((x-\eta)^2,\eta) + F(\eta) + \exp\left(-c_3 \delta^2 \frac{n}{\lambda_n^2} \right). \end{aligned} \tag{40}$$

Consequently from the definition of f, for n sufficiently large

$$\begin{aligned} B_n^*(f,\eta) - f(\eta) &= B_n(f,\eta) - f(\eta) = B_n(F,\eta) - F(\eta) \\ &\leqslant \alpha B_n((x-\eta)^2,\eta) + \exp\left(-c_3 \delta^2 \frac{n}{\lambda_n^2} \right) \\ &= \alpha \frac{4\lambda_n^2}{n} - \alpha \frac{\eta^2}{n} + \exp\left(-c_3 \delta^2 \frac{n}{\lambda_n^2} \right) < -A \frac{\lambda_n^2}{n}, \quad A > 0, \end{aligned}$$

a contradiction because we assumed

$$||w(f - B_n^*(f))|| = o\left(\frac{\lambda_n^2}{n} \right).$$

Similar reasoning leads to a contradiction if $F(\xi) < 0$. □

7. Proof of Theorem 5

If $\omega_2(f,t)_w \leqslant ct^2$, then by Theorem 1 we get $||w[f - B_n^*(f)]|| \leqslant c\lambda_n^2/n$. Now we prove the converse implication. Let $0 \leqslant t \leqslant 1$ be an arbitrary number.

We prove that

$$\max_{0\leqslant h\leqslant t} w(x)|\Delta_h^2 f(x)|\leqslant 2cet^2, \quad |x\pm h|\leqslant h^*, \tag{41}$$

where c is the same constant as in the statement of Theorem 5. This will show that the "main part modulus" in (1) satisfies the requirement.

Suppose that (41) does not hold. We will show that this leads to a contradiction. Indeed, then there exists an $x_0\geqslant 0$ and t_0, h_0 such that

$$w(x_0)\Delta_{h_0}^2 f(x_0) < -2cet_0^2, \quad |x_0\pm h_0|\leqslant h_0^*, \ 0<h_0\leqslant t_0,$$

say. Consider

$$P(x) = -\frac{ce}{w(x_0)}(x-x_0)^2 + \ell(x) + d,$$

where ℓ is the linear function interpolating f at $x_0\pm h_0$, and d is chosen large enough such that $P(x)\geqslant f(x)$ on $[x_0-h_0, x_0+h_0]$. Then (see [2, pp. 138–139])

$$m := \inf\{P(x) - f(x) : x_0 - h_0\leqslant x\leqslant x_0+h_0\}$$

is attained at a point $y\in(x_0-h_0, x_0+h_0)$. Let $P^*(x) = P(x) - m$, then

$$P^*(x)\geqslant f(x), \ x\in[x_0-h_0, x_0+h_0] \quad \text{and} \quad P^*(y) = f(y).$$

Let

$$a' = \max\{x : x\leqslant x_0-h_0, \ P^*(x) = f(x)\},$$

$$b' = \min\{x : x\geqslant x_0+h_0, \ P^*(x) = f(x)\}.$$

Then $a'<y<b'$ and $P^*(x)\geqslant f(x)$ on $[a', b']$.

Let

$$F(x) = \begin{cases} 0, & x\in[a', b'] \\ f(x) - P^*(x), & x\notin[a', b'], \end{cases}$$

so that $f(x)\leqslant P^*(x) + F(x)$, $\forall x\in\mathbb{R}$. Following the proof of Theorem 4 we have

$$|B_n(F, y)|\leqslant \exp\left(-c_1\frac{n}{\lambda_n^2}\right).$$

Thus

$$\begin{aligned} B_n(f, y) - f(y) &\leqslant B_n(P^*+F, y) - P^*(y) = B_n(P^*, y) - P^*(y) + B_n(F, y) \\ &\leqslant -\frac{ce}{w(x_0)}B_n((x-y)^2, y) + \exp\left(-c_1\frac{n}{\lambda_n^2}\right) \\ &= -\frac{ce(4\lambda_n^2 - y)}{w(x_0)n} + \exp\left(-c_1\frac{n}{\lambda_n^2}\right) \leqslant -\frac{3ce\lambda_n^2}{n\omega(x_0)} \end{aligned}$$

for sufficiently large n. On the other hand

$$B_n(f, y) - f(y)\geqslant -c\frac{\lambda_n^2}{nw(y)},$$

whence

$$3e \leqslant \frac{w(x_0)}{w(y)} = e^{Q(y)-Q(x_0)} \leqslant e^{|y-x_0|Q'(x_0+h_0)} \leqslant e^{h_0 Q'(h_0^*)} = e$$

which is a contradiction.

Next we show that the "tail parts" in (1) are also $\leqslant ct^2$. For a fixed $t>0$ define n by $\lambda_n \leqslant t^* < \lambda_{n+1}$. Then

$$t = \frac{1}{Q'(t^*)} \geqslant \frac{1}{Q'(\lambda_{n+1})} = \frac{\lambda_{n+1}}{\sqrt{n+1}} \geqslant c\frac{\lambda_n}{\sqrt{n}}.$$

Since for $x \geqslant \lambda_n$, $B_n^*(f,x)$ is a linear function, we obtain

$$\inf_{\ell \in P_1} ||w(f-\ell)||_{[t^*,\infty)} \leqslant ||w(f-B_n^*(f))||_{[t^*,\infty)}$$
$$\leqslant ||w(f-B_n^*(f))||_{[\lambda_n,\infty)} \leqslant c\frac{\lambda_n^2}{n} \leqslant ct^2.$$

(The case $x \leqslant -\lambda_n$ is analogous.) This shows that the "tail-part" of ω_2 satisfies the stated estimate. □

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